The Goals of Mathematical Education

George Polya

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Part I

I wish to talk to you about the teaching of mathematics in the primary school. In fact my talk will consist of two parts. In the first part I will talk about the aims of teaching mathematics in the primary school. And in the second part, how to teach it.

I must confess that I am talking about these things as an outsider. I was always interested in teaching, but most of my time, about half a century, I taught in the university or various universities. And in the last fifteen years, I was mainly concerned with teaching on the high school level. Thus I am talking to you as an outsider, but you may find one or two points in what I am saying that may be useful to you in your profession.

What is the aim of teaching mathematics in the primary school? It is better to consider the most general question: What is the aim of the schools? And the better question is: What do people generally think is the aim of the schools? The first is the point of view of the parents. Your neighbor Mr. Smith has a son Jimmy. He is against Jimmy being a dropout. He says that if Jimmy drops out from school he will never get a right job. So the aim of the school according to Mr. Smith and all the other Mr. Smiths in the general public is to prepare for a job, to prepare the kids to earn a living. But what is the point of view of the community? It is the same. The community, the country, the state, and the city all want people to earn a living and pay taxes and not live on public assistance. So the community also wants the school to prepare the young people to have a job.

If the parents think a little farther, and the community thinks a little farther, the aim is somewhat changed. Reasonable parents, a reasonable Mr. Smith, wants that his son Jimmy should have a job for which he is well fitted. He will earn more and feel happier. By the way, this is also the aim of the community – that you have jobs on one side and people on the other side and you have to assign to people such jobs that totally they are best fitted

that they produce the greatest output. Or even better, that totally the sum of the happiness should be a maximum. What can the school do for that? The point is that when the kid comes to the school you don't know yet what job will come later, and you don't know for what job he is well fitted, he is best fitted. So what should we do? We should prepare the youngsters so that they can choose between all possible jobs. They must have a view of the whole world around them to recognize for which jobs they will be well fitted. You can express it many ways. I like the following expression: the schools should develop all the interior resources of the child.

We have therefore two kinds of aims in the schools. We have good and narrow aims. The schools should turn out employable adults – adults who can fill a job. But a higher aim is to develop all the resources of the growing child in order that he can fill in the job for which he is best fitted. So the higher aim, I express it so, is to develop all the inner resources of the child.

Now what about mathematics teaching? Mathematics in the primary schools has a good and narrow aim and that is pretty clear in the primary schools. An adult who is completely illiterate is not employable in a modern society. Everybody should be able to read and write and do some arithmetic, and perhaps a little more. Therefore the good and narrow aim of the primary school is to teach the arithmetical skills – addition, subtraction, multiplication, division, and perhaps a little more, as well as to teach fractions, percentages, rates, and perhaps even a little more. Everybody should have an idea of how to measure lengths, areas, volumes. This is a good and narrow aim of the primary schools – to transmit this knowledge – and we shouldn't forget it.

However, we have a higher aim. We wish to develop all the resources of the growing child. And the part that mathematics plays is mostly about thinking. Mathematics is a good school of thinking. But what is thinking? The thinking that you can learn in mathematics is, for instance, to handle abstractions. Mathematics is about numbers. Numbers are an abstraction. When we solve a practical problem, then from this practical problem we must first make an abstract problem. Mathematics applies directly to abstractions. Some mathematics should enable a child at least to handle abstractions, to handle abstract structures. Structure is a fashionable word now. It is not a bad word. I am quite for it.

But I think there is one point which is even more important. Mathematics, you see, is not a spectator sport. To understand mathematics means to be able to do mathematics. And what does it mean doing mathematics? In the first place it means to be able to solve mathematical problems. For the higher aims about which I am now talking are some general tactics of problems – to have the right attitude for problems and to be able to attack

all kinds of problems, not only very simple problems, which can be solved with the skills of the primary school, but more complicated problems of engineering, physics and so on, which will be further developed in the high school. But the foundations should be started in the primary school. And so I think an essential point in the primary school is to introduce the children to the tactics of problem solving. Not to solve this or that kind of problem, not to make just long divisions or some such thing, but to develop a general attitude for the solution of problems.

Part II

Teaching is not a science; it is an art. If teaching were a science there would be a best way of teaching and everyone would have to teach like that. Since teaching is not a science, there is great latitude and much possibility for personal differences. In an old British manual there was the following sentence, "Whatever the subject, what the teacher really teaches is himself." So therefore when I am telling you to teach so or so, please take it in the right spirit. Take as much of my advice as it fits you personally. You must teach yourself.

There are as many good ways of teaching as there are good teachers. But let me tell you what my idea of teaching is. Perhaps the first point, which is widely accepted, is that teaching must be active, or rather active learning. That is the better expression.

You cannot learn just by reading. You cannot learn just by listening to lectures. You cannot learn just by looking at movies. You must add from the action of your own mind in order to learn something. You can call this the Socratic method since Socrates expressed it two thousand years ago very colorfully. He said that the idea should be born in the student's mind and the teacher should just act as a midwife. The idea should be born in the student's mind naturally and the midwife shouldn't interfere too much, too early. But if the labor of birth is too long, the midwife must intervene. This is a very old principle and there is a modern name for it – discovery method. The student learns by his own action. The most important action of learning is to discover it by yourself. This will be the most important part in teaching such that what you discover by yourself will last longer and be better understood.

There are other principles of teaching. If you don't like the word principles, use the words rules of thumb. Learning should be active. Another one was also stated often by all great famous educators – by Socrates, Plato, Comenius, Montessori – and that is that there are certain priorities. For in-

stance, things come before words and so on. This has been stated many times in many forms, but let me quote Kant, who said, "All human cognition begins with intuitions, proceeds hence to conceptions, and ends in ideas." Let me translate this saying into simpler terms. I would say, "Learning begins with action and perception, proceeds hence to words and concepts, and should end in good mental habits."

This is the general aim of mathematics teaching – to develop in each student as much as possible the good mental habits of tackling any kind of problem.

You should develop the whole personality of the student and mathematics teaching should especially develop thinking. Mathematics teaching could also develop clarity and staying power. It could also develop character to some extent but most important is the development of thinking.

My point of view is that the most important part of thinking that is developed in mathematics is the right attitude in tackling problems, in treating problems. We have problems in everyday life. We have problems in science. We have problems in politics. We have problems everywhere. The right attitude to thinking is maybe slightly different from one domain to another, but we have only one head, and therefore it is natural that finally there should be just one method to tackling all kinds of problems. My personal opinion is that the main point in mathematics teaching is to develop the tactics of problem solving.

The two principles of active learning – priority of action and perception – are taken into account by almost all directions in mathematics teaching which are usual today and have some influence.

But perhaps the best developed in the latest time is in Great Britain. There is a foundation called Nuffield Foundation, which propagates this idea of active learning and the priority of action and perception in learning. Their first book has a nice motto. It is allegedly a Chinese proverb that says, "I hear and I forget. I see and I remember. I do and I understand."

So "I hear and I forget." What you just hear you forget quickly. Good advice is very quickly forgotten. What you see with your own eyes is remembered better, but you really understand it when you do it with your own hands. So the motto is "I hear and I forget. I see and I remember. I do and I understand."

Therefore the schools, especially the primary schools, are today in an evolution. A sizable fraction, ten to twenty percent, already have the new method of teaching which can be characterized in the following way in comparison with the old method of teaching. The old method is authoritative and teacher-centered. The new method is permissive and student-centered. In the old time the teacher was in the center of the class or in front of the

class. Everybody looked at him and what he said. Today the individual students should be in the center of the class, and they should be allowed to do whatever good idea comes to their mind. They should be allowed to pursue it in their own way, each by himself or in small groups. If a student has a good idea in class discussion then the teacher changes his plans and enters into the good idea and now the class follows this idea.

I must tell you one name. This is the person who is particularly active in this direction and who is very clever, very good. This is Miss Edith Biggs. She is a particularly gifted teacher who stands in with great enthusiasm and talent for this new permissive and student-centered teaching.

In such a permissive and student-centered class, each group of kids do something else. They play (let's just say that they think that they play, but really they learn). The teacher gives them various materials. A class period consists of the teacher giving kids various materials. They play and they develop their own ideas in play. For instance, one of the materials is squared paper. And a good supply of cubes, cubes of one half inch and several dozens of them, maybe even a hundred. So the kids play with that. It is activity teaching – teaching by action and perception.

Let me give you an example of this activity. The class discusses little rectangles. It should come – that's the main point – from action and perception. It should come from things which kids have seen often enough and touched. So everybody has seen a room, and the walls of an ordinary room are rectangles, or almost rectangles. So you learn what a rectangle is. The floor of the usual room is a rectangle. And any wall is a rectangle. The ceiling is a rectangle. One of the good aims of teaching, then, is to understand length and area. So you measure the length of the rectangles and come to the idea of the perimeter of the rectangles. Then you deal with the area of the rectangle. You build up the rectangle from equal squares, from unit squares, and come to the notion of the area. Anyway, we are now in a class that is somewhat familiar with the area and perimeter of rectangles. On the same sheet of paper, draw overlapping rectangles, with the same perimeter – a perimeter of twenty. It turns out that there are nine such rectangles. They start with width = 1 and height = 9, and then width = 2 and height = 8, and down to width = 9 and height =1.

There are many things to observe – action and perception. Some of the kids will be struck by the observation that all the corners of these rectangles are on a straight line. Then they will notice that one of these rectangles has equal sides and you can ask many questions about it. One of the interesting points is that the teacher should not ask the question but the kids should ask the questions. They all have the same perimeter. Do they have the same area? Which one has the greatest area?

Here is another activity with rectangles. Again take square papers and cut out different rectangles with the same areas, let's say area of 24 square units. Overlap them on the same paper. Now the corners opposite to the one corner in which they overlap are not on a straight line. There is some funny kind of curved line.

Kids with an imagination will join these to make curved lines. So that is another consideration. This is an example of an activity with rectangles where the kids have their own choice. They make their own remarks and the teacher just helps a little now and then with some hints. If the kids have no ideas at all, then the well instructed teacher, who is used to this student-centered teaching, can give a few good hints.

Perhaps one point which Miss Biggs and the Nuffield Foundation do not emphasize sufficiently is the rule of guessing. Guessing comes to us naturally. Everybody tries to guess and does not have to be taught. What has to be taught is reasonable guessing. And especially what has to be taught is to not believe your own guesses but to test them. And students' activity will start much better if you start them by guessing.

Here is one example. One activity is to measure the length and the width of the classroom. Now some kids may be bored by this if they already did it with a former teacher. You can get a little more attention if you start with a guess. You may say, "It seems to me that this classroom is twice as long as it is wide. Is it really?" I hope some of the kids will say, "No, it is longer than twice." Others will say, "No, it is shorter." A very few will say, "Exactly." After they have guessed, they will do the measuring with much more interest because everybody is interested whether his guess will come true or not. This is a very special case in the tactics of problem solving. If you go farther, you will notice that guessing plays an important role. The solution to a problem naturally starts always with a guess – not always with a good guess. On the contrary, usually the guess is never completely good. It is just a little out of center and the art of problem solving consists in great part in correcting your guesses.

I have given you my ideas about how you should teach mathematics. There are the ideas of active learning, the priority of action and perception, and teaching by the activity of the kids to start them by letting them guess. I hope one of these points will find a sympathetic hearing with some of you. Thank you.